Non-Anticommutative Quantum Gravity

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Abstract

A calculation of the one loop gravitational self-energy graph in non-anticommutative quantum gravity reveals that graviton loops are damped by internal momentum dependent factors in the modified propagator and the vertex functions. The non-anticommutative quantum gravity perturbation theory is finite for matter-free gravity and for matter interactions.

1 Introduction

Recently, the consequences for perturbative quantum gravity were investigated, when the gravitational action is given on a noncommutative spacetime geometry, by expanding the metric about a flat Minkowski spacetime and by taking the usual Einstein-Hilbert action, whose fields are functions on ordinary commutative spacetime, except that the products of field quantities are formed by using the Moyal $\star$-product rule [1]. The first order, one loop graviton self-energy was calculated, using a noncommutative action and functional generator $Z[j_{\mu\nu}]$. It was shown that the planar one loop graviton graph and vacuum polarization are essentially the same as for the commutative perturbative result, while the non-planar graviton loop graphs were damped due to the oscillatory behavior of the noncommutative phase factor in the Feynman integrand. Thus, the overall noncommutative perturbative theory remained unrenormalizable and divergent.

In the following, we shall repeat the analysis of perturbative quantum gravity by using a non-anticommutative geometry, defined by the superspace coordinates [2]:

$$\rho^\mu = x^\mu + \beta^\mu, \tag{1}$$

where the x^μ denote the classical commuting c-number coordinates of spacetime, and β^μ denote Grassman coordinates which satisfy

$$\{\beta^\mu, \beta^\nu\} = \beta^\mu\beta^\nu + \beta^\nu\beta^\mu = 0. \tag{2}$$

The familiar commutative coordinates of spacetime are replaced by operators $\hat{\rho}^{\mu}$, which satisfy

$$\{\hat{\rho}^{\mu}, \hat{\rho}^{\nu}\} = 2x^{\mu}x^{\nu} + 2(x^{\mu}\beta^{\nu} + x^{\nu}\beta^{\mu}) - \tau^{\mu\nu}, \tag{3}$$

where $\tau^{\mu\nu}$ is a symmetric two-tensor.

In the non-anticommutative field theory formalism, the product of two operators $\hat{f}$ and $\hat{g}$ has a corresponding $\Diamond$-product

$$(\hat{f}\Diamond\hat{g})(\rho) = \exp\left(-\frac{1}{2}\tau^{\mu\nu}\frac{\partial}{\partial\xi^{\mu}}\frac{\partial}{\partial\zeta^{\nu}}\right)f(\rho+\xi)g(\rho+\zeta)|_{\xi=\zeta=0}$$

$$= f(\rho)g(\rho) - \frac{1}{2}\tau^{\mu\nu}\partial_{\mu}f(\rho)\partial_{\nu}g(\rho) + O(\tau^2), \tag{4}$$

where $\partial_{\mu} = \partial/\partial\rho_{\mu}$.

In Section 2, we shall expand the metric tensor of general relativity about flat Minkowskian spacetime and replace all commutative products of gravitational fields and their derivatives in the Einstein-Hilbert action by $\Diamond$-products in the superspace with coordinates ρ^{μ}. In Section 3, a calculation of the first order, self-energy graviton loop diagrams reveals that they are finite due to the damping of the ultraviolet divergences by the modified graviton propagator. This finite behavior of the loop amplitudes holds for all higher order diagrams. A discussion of the nonlocal nature of the non-anticommutative quantum gravity formalism and the unitarity of graviton amplitudes is given in Section 4, and concluding remarks are made in Section 5.

2   The Non-Anticommutative Gravity Action

We define the non-anticommutative gravitational action in the superspace manifold as

$$S_{\text{grav}} = -\frac{2}{\kappa^2}\int d^4\rho(\sqrt{-g}\Diamond R + 2\sqrt{-g}\lambda), \tag{5}$$

where we use the notation: $\mu, \nu = 0, 1, 2, 3$, $g = \det(g_{\mu\nu})$, the metric signature of Minkowski spacetime is $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, $R = g^{\mu\nu}\Diamond R_{\mu\nu}$ denotes the scalar curvature, λ is the cosmological constant and $\kappa^2 = 32\pi G$ with $c = 1$. The Riemann tensor is defined such that

$$R^{\lambda}{}_{\mu\nu\rho} = \partial_{\rho}\Gamma_{\mu\nu}{}^{\lambda} - \partial_{\nu}\Gamma_{\mu\rho}{}^{\lambda} + \Gamma_{\mu\nu}{}^{\alpha}\Diamond\Gamma_{\rho\alpha}{}^{\lambda} - \Gamma_{\mu\rho}{}^{\alpha}\Diamond\Gamma_{\nu\alpha}{}^{\lambda}. \tag{6}$$

The gravitational action S_{grav} in our non-anticommutative geometry can be rewritten as

$$S_{\text{grav}} = \frac{1}{2\kappa^2} \int d^4\rho [(\mathbf{g}^{\rho\sigma} \Diamond \mathbf{g}_{\lambda\mu} \Diamond \mathbf{g}_{\kappa\nu}$$

$$-\frac{1}{2}\mathbf{g}^{\rho\sigma} \Diamond \mathbf{g}_{\mu\kappa} \Diamond \mathbf{g}_{\lambda\nu} - 2\delta^\sigma_\kappa \delta^\rho_\lambda \mathbf{g}_{\mu\nu}) \Diamond \partial_\rho \mathbf{g}^{\mu\kappa} \Diamond \partial_\sigma \mathbf{g}^{\lambda\nu}], \tag{7}$$

where $\mathbf{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$ and in the following we shall omit the cosmological constant λ. We expand the local interpolating graviton field $\mathbf{g}^{\mu\nu}$ as

$$\mathbf{g}^{\mu\nu} = \eta^{\mu\nu} + \kappa\gamma^{\mu\nu} + O(\kappa^2). \tag{8}$$

Then, for the non-anticommutative superspace

$$\mathbf{g}_{\mu\nu} = \eta_{\mu\nu} - \kappa\gamma_{\mu\nu} + \kappa^2 \gamma_\mu{}^\alpha \Diamond \gamma_{\alpha\nu} - \kappa^3 \gamma_\mu{}^\alpha \Diamond \gamma_{\alpha_\beta} \Diamond \gamma_\nu{}^\beta + O(\kappa^4). \tag{9}$$

Let us consider the non-anticommutative generating functional [3, 4, 5], defined in the superspace manifold:

$$Z[j_{\mu\nu}] = \int d[\mathbf{g}^{\mu\nu}]\Delta[\mathbf{g}^{\mu\nu}] \exp i\left[S_{\text{grav}} + \frac{1}{\kappa} \int d^4\rho \mathbf{g}^{\mu\nu} \Diamond j_{\mu\nu} \right.$$

$$\left. -\frac{1}{\kappa^2\beta} \int d^4\rho \partial_\mu \mathbf{g}^{\mu\nu} \Diamond \partial_\alpha \mathbf{g}^{\alpha\beta} \eta_{\nu\beta} \right], \tag{10}$$

where $(\partial_\mu \mathbf{g}^{\mu\nu} \Diamond \partial_\alpha \mathbf{g}^{\alpha\beta} \eta_{\nu\beta})/\kappa^2\beta$ is the gauge fixing term. Moreover, Δ can be interpreted in terms of fictitious particles and is given by

$$\Delta[\mathbf{g}^{\mu\nu}]^{-1} = \int d[\xi_\lambda]d[\eta_\nu] \exp i\left\{ \int d^4\rho \eta^\nu \Diamond [\eta_{\nu\lambda}\partial^\sigma\partial_\sigma - \kappa(\partial_\lambda\partial^\mu\gamma_{\mu\nu} - \gamma_{\mu\rho} \right.$$

$$\left. \times \eta_{\nu\lambda}\partial^\mu\partial^\rho - \partial^\mu\gamma_{\mu\rho}\eta_{\nu\lambda}\partial^\rho + \partial^\mu\gamma_{\mu\nu}\partial_\lambda)]\Diamond\xi^\lambda \right\}, \tag{11}$$

where ξ^λ and η^λ are the fictitious ghost particle fields.

The gravitational action is expanded as

$$S_{\text{grav}} = S_{\text{grav}}^{(0)} + \kappa S_{\text{grav}}^{(1)} + \kappa^2 S_{\text{grav}}^{(2)} + ... \tag{12}$$

We find the following expanded values of S_{grav}:

$$S_{\text{grav}}^{(0)} = \int d^4\rho (\frac{1}{2}\partial_\sigma\gamma_{\lambda\rho}\Diamond\partial^\sigma\gamma^{\lambda\rho} - \partial_\lambda\gamma^{\rho\kappa}\Diamond\partial_\kappa\gamma^\lambda_\rho - \frac{1}{4}\partial_\rho\gamma\Diamond\partial^\rho\gamma$$

$$-\frac{1}{\alpha}\partial_\rho\gamma^\rho_\lambda\Diamond\partial_\kappa\gamma^{\kappa\lambda}), \tag{13}$$

$$S^{(1)}_{\text{grav}} = \frac{1}{4}\int d^4\rho(-4\gamma_{\lambda\mu}\Diamond\partial^\rho\gamma^{\mu\kappa}\Diamond\partial_\rho\gamma_\kappa{}^\lambda + 2\gamma_{\mu\kappa}\Diamond\partial^\rho\gamma^{\mu\kappa}\Diamond\partial_\rho\gamma$$

$$+2\gamma^{\rho\sigma}\Diamond\partial_\rho\gamma_{\lambda\nu}\Diamond\partial_\sigma\gamma^{\lambda\nu} - \gamma^{\rho\sigma}\Diamond\partial_\rho\gamma\Diamond\partial_\sigma\gamma + 4\gamma_{\mu\nu}\Diamond\partial_\lambda\gamma^{\mu\kappa}\Diamond\partial_\kappa\gamma^{\nu\lambda}), \tag{14}$$

$$S^{(2)}_{\text{grav}} = \frac{1}{4}\int d^4\rho[4\gamma_{\kappa\alpha}\Diamond\gamma^{\alpha\nu}\Diamond\partial^\rho\gamma^{\lambda\kappa}\Diamond\partial_\rho\gamma_{\nu\lambda}+(2\gamma_{\lambda\mu}\Diamond\gamma_{\kappa\nu}-\gamma_{\mu\kappa}\Diamond\gamma_{\nu\lambda})\Diamond\partial^\rho\gamma^{\mu\kappa}\Diamond\partial_\rho\gamma^{\nu\lambda}$$

$$-2\gamma_{\lambda\alpha}\Diamond\gamma_\nu{}^\alpha\Diamond\partial^\rho\gamma^{\lambda\nu}\Diamond\partial_\rho\gamma - 2\gamma^{\rho\sigma}\Diamond\gamma_\nu{}^\kappa\Diamond\partial_\rho\gamma_{\lambda\kappa}\Diamond\partial_\sigma\gamma^{\nu\lambda}$$

$$+\gamma^{\rho\sigma}\Diamond\gamma^{\nu\lambda}\Diamond\partial_\sigma\gamma_{\nu\lambda}\Diamond\partial_\rho\gamma - 2\gamma_{\mu\alpha}\Diamond\gamma^{\alpha\nu}\Diamond\partial^\lambda\gamma^{\mu\kappa}\Diamond\partial_\kappa\gamma_{\nu\lambda}], \tag{15}$$

where $\gamma = \gamma^\alpha{}_\alpha$.

The free part of the action is not the same as the commutative case. However, we can choose to quantize the field quantities $\gamma_{\mu\nu}$ with the same vacuum state as in the commutative case [2]. In particular, the measure in the functional integral formalism is the same as the commutative theory, for in momentum space the additional factors due to the use of the $\Diamond$-product disappear when we impose the normalization condition for the partition function.

The modified graviton propagator in the fixed gauge $\beta = -1$ is given in superspace by [2]

$$i\bar{D}^{\text{grav}}_{\mu\nu\rho\sigma}(\rho - \rho') = (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma})$$

$$\times\frac{i}{(2\pi)^4}\int \frac{d^4p\exp[\frac{1}{2}(p\tau p)]}{p^2 + i\epsilon}\exp[ip(\rho - \rho')], \tag{16}$$

where $(p\tau p) = p_\mu\tau^{\mu\nu}p_\nu$. In momentum space this becomes

$$i\bar{D}^{\text{grav}}_{\mu\nu\rho\sigma}(p) = (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma})\frac{i\exp[\frac{1}{2}(p\tau p)]}{p^2 + i\epsilon}. \tag{17}$$

The ghost propagator in momentum space is

$$i\bar{D}^G_{\mu\nu}(p) = \frac{\eta_{\mu\nu}i\exp[\frac{1}{2}(p\tau p)]}{p^2 + i\epsilon}. \tag{18}$$

In momentum space, a graviton interaction diagram has an additional factor which takes the form [2]:

$$V(q_1, q_2..., q_n) = \exp\left(\frac{1}{2}\sum_{i<j}q_i \cdot q_j\right), \tag{19}$$

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where

$$q_i \cdot q_j \equiv q_{i\mu}\tau^{\mu\nu}q_{j\nu}. \tag{20}$$

In flat spacetime, the only changes of the Feynman rules consist of inserting a modified graviton propagator $\bar{D}^{\text{grav}}_{\mu\nu\alpha\beta}$ in every internal line and inserting a factor $V(q_1, ..., q_n)$ in every diagram.

Let us consider the effects of an infinitesimal gauge transformation

$$\rho^{'\mu} = \rho^\mu + \zeta^\mu \tag{21}$$

on the non-anticommutative generating functional $Z[j_{\mu\nu}]$, where ζ^μ can depend on ρ^μ and $\gamma^{\mu\nu}$. We get

$$\delta\mathbf{g}^{\mu\nu}(\rho) = -\zeta^\lambda(\rho)\lozenge\partial_\lambda\mathbf{g}^{\mu\nu}(\rho) + \partial_\rho\zeta^\mu(\rho)\lozenge\mathbf{g}^{\rho\nu}(\rho)$$

$$+\partial_\sigma\zeta^\nu(\rho)\lozenge\mathbf{g}^{\mu\sigma}(\rho) - \partial_\alpha\zeta^\alpha(\rho)\lozenge\mathbf{g}^{\mu\nu}(\rho). \tag{22}$$

We now find that

$$\delta\gamma_{\mu\nu}(\rho) = -\zeta^\lambda\lozenge\partial_\lambda\gamma_{\mu\nu} + \partial^\rho\zeta_\mu\lozenge\gamma_{\rho\nu} + \partial^\rho\zeta_\nu\lozenge\gamma_{\mu\rho}$$

$$-\partial^\rho\zeta_\rho\lozenge\gamma_{\mu\nu} + \frac{1}{\kappa}(\partial_\nu\zeta_\mu + \partial_\mu\zeta_\nu - \partial^\rho\zeta_\rho\eta_{\mu\nu}). \tag{23}$$

The functional generator Z is invariant under changes in the integration variable and the transformation (23).

The non-anticommutative gauge transformations (23) should be considered part of an $NACSO(3,1)$ group of gauge transformations. It is clear that in the limit $\beta^\mu \to 0$ and $|\tau^{\mu\nu}| \to 0$, the standard local Lorentz group of gauge transformations $SO(3,1)$ is recovered.

3   Gravitational Self-Energy

The lowest order contributions to the graviton self-energy will include the standard graviton loops, the ghost field loop contributions and the measure loop contributions. In perturbative gravity theory, the first order vacuum polarization tensor $\Pi^{\mu\nu\rho\sigma}$ must satisfy the Slavnov-Ward identities [5]:

$$p_\mu p_\rho \bar{D}^{\mu\nu\alpha\beta}(p)\Pi_{\alpha\beta\gamma\delta}(p)\bar{D}^{\gamma\delta\rho\sigma}(p) = 0. \tag{24}$$

The basic lowest order graviton self-energy diagram is determined by

$$\Pi_{\mu\nu\rho\sigma}(p) = \frac{1}{2}\kappa^2 \int d^4q \,\mathcal{U}_{\mu\nu\alpha\beta\gamma\delta}(p, -q, q-p)\bar{D}^{\text{grav}\,\alpha\beta\kappa\lambda}(q)$$

$$\times \bar{D}^{\text{grav}\, \gamma\delta\tau\xi}(p-q)\mathcal{U}_{\kappa\lambda\tau\xi\rho\sigma}(q,p-q,-p)V(q,p-q,p), \qquad (25)$$

where $\mathcal{U}$ is the three-graviton vertex function

$$\mathcal{U}_{\mu\nu\rho\sigma\delta\tau}(q_1,q_2,q_3) = -\frac{1}{2}\Big[q_{2(\mu}q_{3\nu)}\Big(2\eta_{\rho(\delta}\eta_{\tau)\sigma} - \eta_{\rho\sigma}\eta_{\delta\tau}\Big)$$

$$+q_{1(\rho}q_{3\sigma)}\Big(2\eta_{\mu(\delta}\eta_{\tau)\nu} - \eta_{\mu\nu}\eta_{\delta\tau}\Big) + ...\Big], \qquad (26)$$

and the ellipsis denote similar contributions. We must add to this result the contributions from the one loop fictitious ghost particle graph and tadpole graph.

The modified graviton propagator has the asymptotic behavior as $q^2 \to \infty$:

$$\bar{D}^{\text{grav}} \sim \exp\Big[\frac{1}{2}(q\tau q)\Big]. \qquad (27)$$

If we choose an orthonormal frame such that $\tau^{\mu\nu} = \eta^{\mu\nu}/\Lambda_{\text{grav}}^2$, then we get [2]

$$\bar{D}^{\text{grav}} \sim \exp\Big(\frac{1}{2}q^2/\Lambda_{\text{grav}}^2\Big). \qquad (28)$$

We can now perform an analytic continuation in the invariant momentum q such that $q^2 = -k^2$ with $k^2 = k_4^2 + k_1^2 + k_2^2 + k_3^2 > 0$. Then, we have in Euclidean momentum space

$$\bar{D}^{\text{grav}} \sim \exp\Big(-\frac{1}{2}k^2/\Lambda_{\text{grav}}^2\Big). \qquad (29)$$

The asymptotic behavior in Euclidean momentum space of the modified graviton propagator will damp out the ultraviolet behavior of the integrands in the graviton one loop self-energy diagrams. Thus, these diagrams lead to a finite lowest order vacuum polarization in non-anticommutative quantum gravity. Higher order graviton self-energy loops will contain products of the modified propagator, corresponding to the number of internal lines in a loop diagram, which will also damp out the ultraviolet behavior of the internal momentum integrations. It follows that the higher order pure gravity loops and loops occurring in gravity-matter interactions will be finite.

In order to retain physical behavior of graviton scattering amplitudes and crossing symmetry relations, and avoid essential singularities at infinite momentum, we are required to choose an orthonormal frame with $\tau^{00} =$

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$\tau^{0n} = 0$ and $\tau^{mn} = -\delta^{mn}/\Lambda^2_{\text{grav}}$ $(m, n = 1, 2, 3)$ [6]. It follows that for $\mathbf{q}^2 > 0$ (where $\mathbf{q}$ denotes the three-momentum vector) , we have for $q^2 \to \pm\infty$:

$$\bar{D}^{\text{grav}} \sim \exp\Big(-\frac{1}{2}\mathbf{q}^2/\Lambda^2_{\text{grav}}\Big). \tag{30}$$

4 Nonlocality and Unitarity

The infinite derivatives that occur in the $\diamond$-product of fields render the non-anticommutative field theories nonlocal. This is also true for noncommutative quantum field theory [7, 8, 9] and noncommutative quantum gravity [1, 10]. However, with the choice $\tau^{00} = \tau^{n0} = 0, \tau^{mn} \neq 0$, we can suppress some of the potentially unphysical acausal behavior of the non-anticommutative amplitudes and retain some of the standard features of quantum field theory such as the canonical Hamiltonian formalism. But if we consider a non-perturbative treatment of quantum gravity in a non-anticommutative geometry, then standard perturbative field theory methods do not apply and it is possible that one must view the nonlocal dynamics in a new way. Only future developments in non-perturbative quantum gravity, and, in particular, the way such developments will affect calculations using non-anticommutative geometry, can provide us with deeper insights into these problems.

An analysis of the unitarity of amplitudes in scalar non-anticommutative field theory [6], showed that it can be satisfied, because the amplitudes in non-anticommutative field theory are only modified by *entire* functions, which do not introduce any new unphysical singularities in a finite region of the complex momentum plane. Moreover, with the choice $\tau^{00} = \tau^{n0} = 0, \tau^{mn} \neq 0$, the scattering amplitudes are regular at infinite energies and crossing symmetry relations retain their physical behavior. These results can be extended to non-anticommutative quantum gravity. This choice of the symmetric tensor $\tau^{\mu\nu}$ breaks local Lorentz invariance. However, it is possible to break the Lorentz symmetry 'softly' by a spontaneous symmetry breaking mechanism, which involves adding a Higgs breaking mechanism contribution to the action that uses a Higgs vector to break local $SO(3,1)$ to $O(3)$ at the small distance ℓ_{grav} ($\ell_{\text{grav}} \sim 1/\Lambda_{\text{grav}}$) when quantum gravity is expected to become important [11].

5   Conclusions

We have developed a perturbative, non-anticommutative quantum gravity formalism by using the $\diamondsuit$-product in the gravity action, wherever products of gravitational fields and their derivatives occur in the superspace manifold. By expanding about Minkowski flat spacetime, we were able to calculate the loop graphs to first order. We find that by using the Feynman rules appropriate for non-anticommutative quantum field theory, the loop graphs are finite to first order and to all orders due to the Gaussian damping of the modified graviton propagator. Since the loop graphs are finite, we do not find any peculiar ultraviolet/infrared behavior as is encountered in noncommutative field theory [7].

From these results, we expect that non-anticommutative Yang-Mills gauge theories with the action defined in superspace

$$S_{\mathrm{YM}} = -\frac{1}{4} \int d^4 \rho F^{a\mu\nu} \diamondsuit F_{a\mu\nu}, \tag{31}$$

where

$$F^a_{\mu\nu} = \partial_\nu A^a_\mu - \partial_\mu A^a_\nu - \epsilon^{abc} A^b_\mu \diamondsuit A^c_\nu, \tag{32}$$

will also yield finite loop diagrams with an energy scale parameter Λ_{YM}.

Acknowledgments

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